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LETTER TO THE EDITOR

Singularity spectrum of a fractal diagram for a Hamiltonian system

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Abstract. We determine numerically the relevant $f(\alpha)$ spectrum of scaling indices for the fractal diagram of the standard map. Infinite partitions, related by the Gauss transformation and an appropriate measure had to be used in order to obtain convergent results. This choice of the measure and the partitions is motivated by the method of modular smoothing.

Fractal diagrams and critical functions describe the stability of periodic orbits and breakup of invariant tori in typical Hamiltonian systems with two degrees of freedom. These are complicated fractal functions of a frequency (or a frequency ratio), with a non-trivial, globally self-similar structure. The fractal structure of these objects is related to the famous problem of small denominators, and to the problem of approximating irrational numbers by rationals. Usually, the global scaling of fractals originating in similar dynamical problems is described by the corresponding $f(\alpha)$ and other 'thermodynamic' functions [1]. It is the purpose of this letter to report our calculations of an appropriate $f(\alpha)$ function for the fractal diagram of a typical area-preserving map, i.e. the standard map given by the following equations:

$$\begin{aligned} r_{n+1} &= r_n - (k/2\pi)\sin(2\pi\theta_n) \\ \theta_{n+1} &= \theta_n + r_{n+1} \quad \bmod 1\,. \end{aligned} \tag{1}$$

On the other hand, global scaling of fractal diagrams and critical functions of typical Hamiltonian systems with two degrees of freedom is also explored and used in the theory of modular smoothing [2]. In our construction of the function $f(\alpha)$ for the fractal diagram we were guided by a result from the theory of modular smoothing, which indicates the scaling function appropriate to this problem [3].

Let us first briefly recapitulate definitions of the fractal diagram [4] and the critical function [5]. The fractal diagram for the standard map, denoted by $k(m/n)$, is a function of the frequency, related to periodic orbits, such that a periodic orbit with frequency m/n is stable (elliptic) when the parameter k satisfies $k < k(m/n)$, and unstable (hyperbolic) when $k > k(m/n)$. The critical function $K(\nu)$ is also a function of the frequency, related to invariant circles, such that if $k < K(\nu)$ a quasiperiodic orbit with frequency ν fills a smooth invariant circle. If $k > K(\nu)$ then there is no smooth invariant circle filled by quasiperiodic

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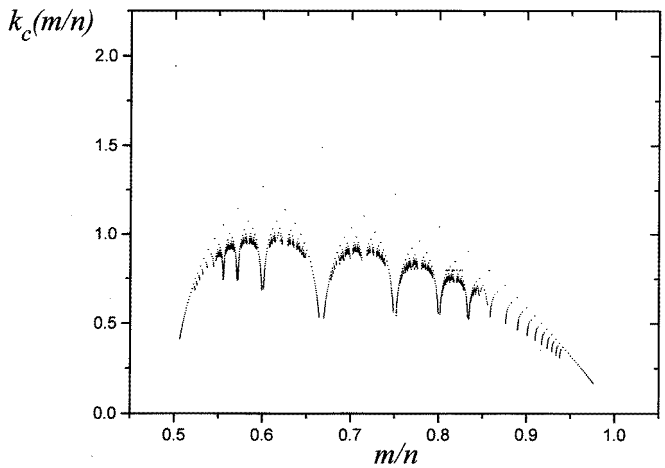

Figure 1. Fractal diagram for the standard map.

orbits with frequency ν. The two functions are complicated fractal objects, and are related by the Greene conjecture [6]

$$\lim_{m/n\to\nu} k(m/n) = K(\nu)\,. \tag{2}$$

The fractal diagram $k(m/n)$ for the standard map is illustrated in figure 1.

The full complexity of scaling of the fractal diagram can be described by a spectrum of critical exponents α of an appropriate measure, and by their densities $f(\alpha)$. However, our construction of the $f(\alpha)$ function differs from the applications of the formalism in most cases of fractals embedded in one-dimensional sets. In our case the support of the measure is the full interval [0, 1], but the measure itself has a fractal density determined by $k(m/n)$. The function $f(\alpha)$ is then a density of points in [0, 1] where $k(m/n)$ has singularities with the same exponent.

A sequence of partitions of the interval [0, 1] is defined using the Gauss transformation: $x \to G(x) \equiv x' = 1/x - \{1/x\}$, where $\{1/x\}$ denotes the integer part of $1/x$. The zero level partition is the interval [0, 1] itself. The first level partition contains all intervals of the form $[1/n, 1/n-1]$, $n = 2, 3 \ldots$. The intervals in the second level partition are obtained by applying the inverse of the Gauss transformation on the boundary points of the intervals in the first level partition. Each interval $[1/n, 1/n-1]$ at the first level gives an infinite number of intervals of the second level partition. Thus each partition (except at the zero level) has an infinite number of elements (intervals), and intervals of the $(i-1)$th level partition are related to the intervals of the ith level partition by the Gauss transformation. The partition function is now defined by the following infinite sum:

$$\Gamma_i(q,\tau) = \sum_j \frac{p_j^q}{l_j^\tau} \tag{3}$$

where

$$p_j = |k(m_j/n_j) - k(m_{j-1}/n_{j-1})|^{n_j} \tag{4}$$

and

$$l_j = |m_j/n_j - m_{j-1}/n_{j-1}|\,. \tag{5}$$

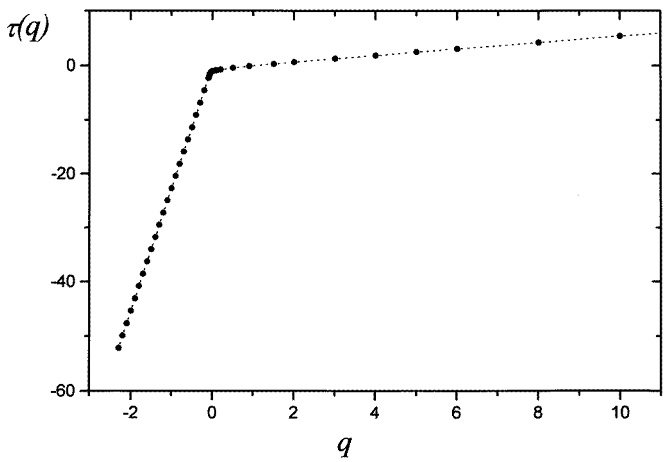

Figure 2. The function $\tau(q)$.

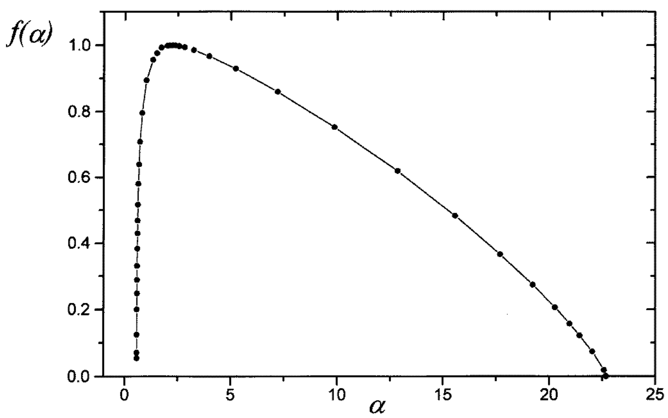

Figure 3. The spectrum of scaling indices $f(\alpha)$ for the fractal diagram.

m_j/n_j and m_{j-1}/n_{j-1} are the boundary points of the jth interval of the ith level partition. The exponents q and τ are determined from the conditions that the infinite sums are convergent and that $\Gamma_i(q,\tau) \to 1$ for large enough i. Naturally, this is possible only if the measure is chosen appropriately.

In practice the infinite sums $\Gamma_i(q,\tau)$ have to be truncated in such a way that the value of $\tau(q)$ which makes $\Gamma_i(q,\tau) = 1$ is not changed by adding the first neglected terms in the sums. Our calculations always satisfied this requirement.

Our choice of the measure (4) and the partitions related by the Gauss transformation is motivated by a result from the theory of modular smoothing. One of the main results in this theory is that the function

$$l_1(m/n) = \ln k(m/n) - \frac{m}{n} \ln k(G(m/n)) \tag{6}$$

where $G(m/n)$ denotes the Gauss transformation of m/n, is continuous, despite the fact that $k(m/n)$ is an everywhere discontinuous function. Thus $k(m/n)^n/k(G(m/n))^m$ is a continuous function which describes the scaling of the fractal diagram with respect to the Gauss transformation. Indeed, some other measures, like for example $p_j = |k(m_j/n_j) - k(m_{j-1}/n_{j-1})|$, or other partitions, like for example the one given by the Farey tree, do not give convergent results.

Once the function $\tau(q)$ is calculated, the function $f(\alpha)$ is given by the Legendre transformation of $\tau(q)$,

$$\alpha(q) = \frac{\mathrm{d}\tau}{\mathrm{d}q} \qquad f(q) = q\alpha(q) - \tau\,. \tag{7}$$

Eliminating q gives the function $f(\alpha)$. Our numerical results are presented in figures 2 and 3. The spectrum of the corresponding generalized dimensions $D_q = (q-1)\tau$ is given in figure 4. The convergence is visually better illustrated using D_q's from the successive levels rather than the corresponding $\tau(q)$. Let us stress once more that only with the choice of the measure given by (4) and the partitions related by the Gauss transformation were we

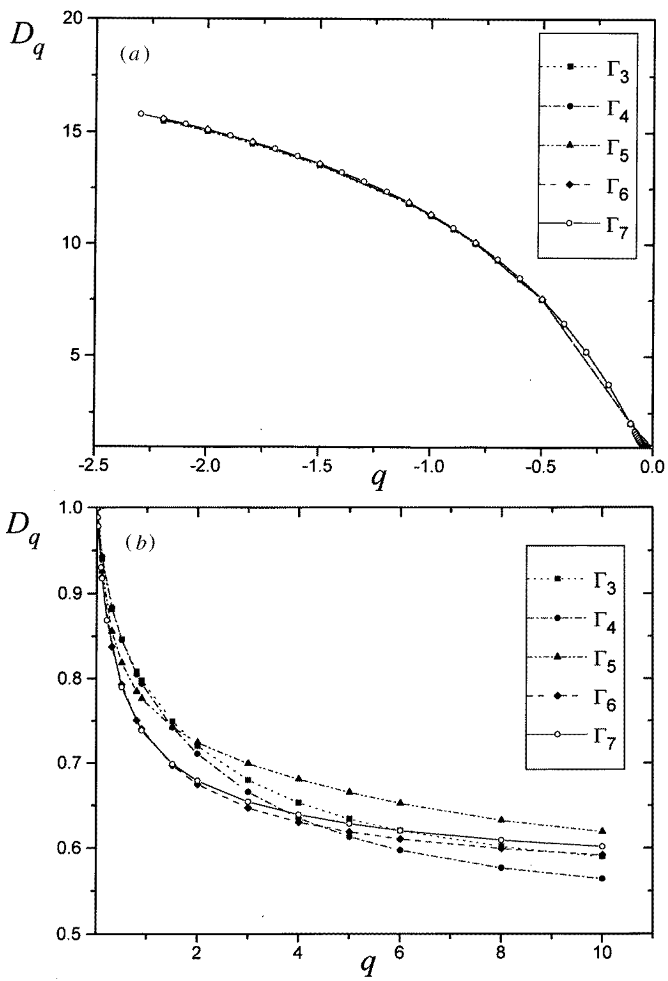

Figure 4. (*a*), (*b*) The figures illustrate the convergence of the spectrum of generalized dimensions D_q for the fractal diagram. Shown are the values of D_q determined from Γ_i at levels $i = 3, 4, 5, 6, 7$. The D_q curves for negative q are almost indistinguishable.

able to obtain the convergent results. It would be interesting to examine more closely the relation between the functions $l_1(m/n)$ and $f(\alpha)$.

Finally, we believe that our results for the fractal diagram of the standard map are universal for a class of Hamiltonian systems with two degrees of freedom [7].

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